

The inverse along a product and its applications

Huihui Zhu^{a,b}, Pedro Patrício^b, Jianlong Chen^{a,*}

^a*Department of Mathematics, Southeast University, Nanjing 210096, China.*

^b*CMAT-Centro de Matemática, Universidade do Minho, Braga 4710-057, Portugal.*

Abstract

In this paper, we study the recently defined notion of the inverse along an element. An existence criterion for the inverse along a product is given in a ring. As applications, we present the equivalent conditions for the existence and expressions of the inverse along a matrix.

Keywords:

Von Neumann regularity, Inverse along an element, Green's relations, Matrices over a ring

2010 MSC: 15A09, 20H25

1. Introduction

In this paper, R is an associative ring with unity 1. An element $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such x is called an inner inverse of a , denoted by a^- . An arbitrary inner inverse of a is denoted by $a^{(1)}$. We call b an outer inverse of a provided that $bab = b$. If b is both an inner and an outer inverse of a , then it is a reflexive inverse of a , and is denoted by a^+ .

Given a semigroup S , S^1 denotes the monoid generated by S . Following Green [1], Green's preorders and relations in a semigroup are defined by

$$a \leq_{\mathcal{L}} b \Leftrightarrow S^1 a \subset S^1 b \Leftrightarrow \text{there exists } x \in S^1 \text{ such that } a = xb.$$

$$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subset bS^1 \Leftrightarrow \text{there exists } x \in S^1 \text{ such that } a = bx.$$

$$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b.$$

$$a \mathcal{L} b \Leftrightarrow S^1 a = S^1 b \Leftrightarrow \text{there exist } x, y \in S^1 \text{ such that } a = xb \text{ and } b = ya.$$

*Corresponding author

Email addresses: ahzh08@sina.com (Huihui Zhu), pedro@math.uminho.pt (Pedro Patrício), jlchen@seu.edu.cn (Jianlong Chen)

$a\mathcal{R}b \Leftrightarrow aS^1 = bS^1 \Leftrightarrow$ there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$.
 $a\mathcal{H}b \Leftrightarrow a\mathcal{L}b$ and $a\mathcal{R}b$.

Recently, Mary [4] introduced the notion of the inverse along an element that is based on Green's relation in a semigroup S . Given $a, d \in S$, an element $a \in S$ is invertible along d [4] if there exists b such that $dab = d = bad$ and $b \leq_{\mathcal{H}} d$. The element b above is unique if it exists, and is denoted by $a^{\parallel d}$. Recall that $a^{\parallel d}$ exists implies that d is regular. Later, Mary and Patrício [5] proved that a is invertible along d if and only if $d\mathcal{H}dad$, which gave a new existence criterion for the inverse along an element. Further, given a regular element d , they [5, 6] characterized the existence of $a^{\parallel d}$ by means of a unit and d^- in a ring. Moreover, the representation of $a^{\parallel d}$ is given. As applications, they [6] derived the equivalent conditions for the existence and the formula of the inverse along a regular lower triangular matrix. More results on the inverse along an element can be found in mathematical literature [3, 9].

Motivated by papers [5, 6], we investigate the inverse along a product pmq (m is regular) in a ring, extending the results in [5, 6]. As applications, the inverse along a regular matrix $\begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$ is given under some conditions.

2. The inverse along a product pmq

In this section, we begin with some lemmas which play important roles in the sequel.

Lemma 2.1. *Given $a, b \in R$, then $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 2.1 is known as the Jacobson's Lemma (see e.g. [2]).

Lemma 2.2. ([8, Theorem 1]) *Let R be a ring and e an idempotent in R . Then $exe + 1 - e$ is invertible in R if and only if exe is invertible in eRe .*

The next theorem, a main result of this paper, gives an existence criterion of the inverse along a product pmq in a ring.

Theorem 2.3. *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent*

- (i) *a is invertible along pmq .*
- (ii) *$u = mqap + 1 - mm^{(1)}$ is invertible.*

(iii) $v = qapm + 1 - m^{(1)}m$ is invertible.

In this case,

$$a^{\parallel pmq} = pu^{-1}mq = pmv^{-1}q.$$

PROOF. It follows from Lemma 2.1 that $(ii) \Leftrightarrow (iii)$. Next, it is sufficient to prove $(i) \Leftrightarrow (ii)$.

$(i) \Rightarrow (ii)$ Suppose that a is invertible along pmq . From $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then there exist p' and q' such that $p'pm = m = mqq'$. In view of [5, Theorem 2.2], we know that a is invertible along pmq if and only if $pmq\mathcal{H}pmqapmq$. There are $x, y \in R$ such that

$$pmq = xpmqapmq = pmqapmqy. \quad (1)$$

Multiplying the above equation (1) by p' on the left yields

$$mq = mqapmqy.$$

Multiplying the above equation (1) by q' on the right yields

$$pm = xpmqapm.$$

Hence,

$$mqapmm^{(1)}(mqyq'm^{(1)}mm^{(1)}) = mm^{(1)} = (mm^{(1)}p'xpm^{(1)})mqapmm^{(1)}.$$

The equalities above show that $mqapmm^{(1)}$ is invertible in $mm^{(1)}Rmm^{(1)}$. By Lemma 2.2, $mqapmm^{(1)} + 1 - mm^{(1)}$ is invertible in R . Again, Lemma 2.1 ensures that $u = mqap + 1 - mm^{(1)}$ is invertible.

$(ii) \Rightarrow (i)$ Suppose that u , therefore v are invertible. From $um = mv = mqapm$, it follows that $pmq = pu^{-1}mqapmq = pmqapmv^{-1}q$ and $pu^{-1}mq = pmv^{-1}q$. Pose $b = pu^{-1}mq = pmv^{-1}q$, then $b \leq_{\mathcal{H}} pmq$ since $pu^{-1}mq = pu^{-1}p'pmq = pmqq'v^{-1}q$.

Hence, a is invertible along pmq . Moreover,

$$a^{\parallel pmq} = pu^{-1}mq = pmv^{-1}q.$$

The proof is completed. \square

If p is left invertible and q is right invertible, then $m\mathcal{L}pm$ and $m\mathcal{R}mq$. As a special result of Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $p, a, q, m \in R$ with m regular. If p is left invertible and q is right invertible, then the following conditions are equivalent*

- (i) *a is invertible along pmq .*
- (ii) *$u = mqap + 1 - mm^{(1)}$ is invertible.*
- (iii) *$v = qapm + 1 - m^{(1)}m$ is invertible.*

In this case,

$$a^{\parallel pmq} = pu^{-1}mq = pmv^{-1}q.$$

Taking $p = q = 1$, we get

Corollary 2.5. ([5, Theorem 3.2] and [6, Theorem 1.3]) *Let m be a regular element of a ring R . Then the following are equivalent*

- (i) *a is invertible along m .*
- (ii) *$u = ma + 1 - mm^{(1)}$ is invertible.*
- (iii) *$v = am + 1 - m^{(1)}m$ is invertible.*

In this case,

$$a^{\parallel m} = u^{-1}m = mv^{-1}.$$

3. Applications to the inverse along a matrix

Mary, Patrício [6] gave some equivalent conditions for the existence of the inverse along a regular lower triangular matrix $\begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix}$ over a Dedekind-finite ring. It would be interesting to find the related existence criteria and formula of the inverse along a regular matrix $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$, in the general case.

By $R_{2 \times 2}$ we denote the ring of 2×2 matrices over R . Let $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2 \times 2}$ and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: PMQ$. Given a lower triangular matrix $M = \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix}$ with d_2 and d_3 regular, Patrício and Puystjens [7] proved that M is regular if and only if $w = (1 - d_3d_3^+)d_1(1 - d_2^+d_2)$ is regular. In this case,

$$MM^- = \begin{bmatrix} d_2d_2^+ & 0 \\ (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+ & d_3d_3^+ + ww^-(1 - d_3d_3^+) \end{bmatrix}.$$

First, we consider the inverse along a regular $(2, 2, 0)$ matrix in a ring.

Theorem 3.1. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2 \times 2}$ with d_3 and d_3 regular. If $c^{\parallel d_2}$ exists, then $A^{\parallel D}$ exists if and only if $\xi = \beta - \alpha c^{\parallel d_2} a$ is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\parallel d_2}) & \xi^{-1}d_3 \\ c^{\parallel d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\parallel d_2})] & -c^{\parallel d_2}a\xi^{-1}d_3 \end{bmatrix}$, where

$$\begin{aligned} \alpha &= d_1c + d_3d - (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+, \\ \beta &= d_1a + d_3b + (1 - ww^-)(1 - d_3d_3^+), \\ \xi &= \beta - \alpha c^{\parallel d_2}a. \end{aligned}$$

PROOF. We have $MAP = \begin{bmatrix} d_2c & d_2a \\ d_1c + d_3d & d_1a + d_3b \end{bmatrix}$. Hence,

$$U = MAP + I - MM^- = \begin{bmatrix} u & d_2a \\ \alpha & \beta \end{bmatrix}, \text{ where}$$

$$\begin{aligned} u &= d_2c + 1 - d_2d_2^+, \\ \alpha &= d_1c + d_3d - (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+, \\ \beta &= d_1a + d_3b + (1 - ww^-)(1 - d_3d_3^+). \end{aligned}$$

Since $c^{\parallel d_2}$ exists, it follows that $u = d_2c + 1 - d_2d_2^+$ is invertible and $c^{\parallel d_2} = u^{-1}d_2$. Using Schur complements we get the factorization

$$U = \begin{bmatrix} 1 & 0 \\ \alpha u^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1 & c^{\parallel d_2}a \\ 0 & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha c^{\parallel d_2}a$. Hence, U is invertible if and only if ξ is invertible.

Note that $U^{-1} = \begin{bmatrix} 1 & -c^{\parallel d_2}a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix}$. Then

$$A^{\parallel D} = PU^{-1}M = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\parallel d_2}) & \xi^{-1}d_3 \\ c^{\parallel d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\parallel d_2})] & -c^{\parallel d_2}a\xi^{-1}d_3 \end{bmatrix}.$$

The proof is completed. \square

Now, suppose that d_4 in the matrix D is regular and set $e = 1 - d_4 d_4^+$, $f = 1 - d_4^+ d_4$ and $s = d_1 - d_3 d_4^+ d_2$. We have the following decomposition

$$D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3 d_4^+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & d_3 f \\ e d_2 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^+ d_2 & 1 \end{bmatrix} =: PMQ.$$

We next discuss the inverse of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ along a regular matrix D , under certain conditions.

Theorem 3.2. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2 \times 2}$ with d_4 regular. With the notations above, if $d_3 f = 0$ and $a^{\parallel s}$ exists, then $A^{\parallel D}$ exists if and only if $\xi = \beta - \alpha a^{\parallel s}(ad_3 d_4^+ + c)$ is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\parallel s}) & \xi^{-1} d_4 \end{bmatrix}$, where

$$\begin{aligned} u &= sa + 1 - ss^+, \\ t &= ed_2(1 - s^+ s), \\ \alpha &= d_2 a + d_4 b - (1 - tt^-)ed_2 s^+, \\ \beta &= (d_2 a + d_4 b)d_3 d_4^+ + d_2 c + d_4 d + (1 - tt^-)e, \\ \xi &= \beta - \alpha a^{\parallel s}(ad_3 d_4^+ + c), \\ x_1 &= [(1 - a^{\parallel s} a)d_3 d_4^+ - a^{\parallel s} c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1 \alpha u^{-1}. \end{aligned}$$

PROOF. If $d_3 f = 0$, then $M = \begin{bmatrix} s & 0 \\ ed_2 & d_4 \end{bmatrix}$. Note that the regularity of D is equivalent to the regularity of M . Hence, it follows from [7, Theorem 1] that

$$I - MM^- = \begin{bmatrix} 1 - ss^+ & 0 \\ -(1 - tt^-)ed_2 s^+ & (1 - tt^-)e \end{bmatrix},$$

where $t = ed_2(1 - s^+ s)$.

Note that $MQAP = \begin{bmatrix} sa & s(ad_3 d_4^+ + c) \\ d_2 a + d_4 b & (d_2 a + d_4 b)d_3 d_4^+ + d_2 c + d_4 d \end{bmatrix}$. We have

$$U = MQAP + I - MM^- = \begin{bmatrix} u & s(ad_3 d_4^+ + c) \\ \alpha & \beta \end{bmatrix},$$

where

$$\begin{aligned} u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b - (1 - tt^-)ed_2s^+, \\ \beta &= (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d + (1 - tt^-)e. \end{aligned}$$

In this case,

$$U^{-1} = \begin{bmatrix} 1 & -a^{\parallel s}(ad_3d_4^+ + c) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha a^{\parallel s}(ad_3d_4^+ + c)$.

By calculations, $A^{\parallel D} = PU^{-1}MQ = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\parallel s}) & \xi^{-1}d_4 \end{bmatrix}$, where

$$\begin{aligned} x_1 &= [(1 - a^{\parallel s}a)d_3d_4^+ - a^{\parallel s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

The proof is completed. \square

If d_4 is invertible, then $e = f = 0$. Hence, we have the following corollary.

Corollary 3.3. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2 \times 2}$ with d_4 invertible. If $a^{\parallel s}$ exists, then $A^{\parallel D}$ exists if and only if $\xi = \beta - \alpha a^{\parallel s}(ad_3d_4^{-1} + c)$ is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\parallel s}) & \xi^{-1}d_4 \end{bmatrix}$, where

$$\begin{aligned} s &= d_1 - d_3d_4^{-1}d_2, \\ u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b, \\ \beta &= \alpha d_3d_4^{-1} + d_2c + d_4d, \\ \xi &= \beta - \alpha a^{\parallel s}(ad_3d_4^{-1} + c), \\ x_1 &= [(1 - a^{\parallel s}a)d_3d_4^{-1} - a^{\parallel s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

In Theorem 3.2, take $d_3 = 0$, then $s = d_1$. We can get the formula and equivalence for the existence of the inverse along a regular lower triangular matrix obtained in [6].

Corollary 3.4. ([6, Theorem 3.1]) *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix} \in R_{2 \times 2}$ with d_4 regular. With the notations above, if $a^{\parallel d_1}$ exists, then $A^{\parallel D}$ exists if and only if $\xi = \beta - \alpha a^{\parallel d_1} c$ is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} a^{\parallel d_1} & -a^{\parallel d_1} c \xi^{-1} d_4 \\ \xi^{-1}(d_2 - \alpha a^{\parallel d_1}) & \xi^{-1} d_4 \end{bmatrix}$, where

$$\begin{aligned} u &= d_1 a + 1 - d_1 d_1^+, \\ t &= e d_2 (1 - d_1^+ d_1), \\ \alpha &= d_2 a + d_4 b - (1 - t t^-) e d_2 d_1^+, \\ \beta &= d_2 c + d_4 d + (1 - t t^-) e, \\ \xi &= \beta - \alpha a^{\parallel d_1} c. \end{aligned}$$

By taking $e d_2 = 0$ in Theorem 3.2, we get the following corollary.

Corollary 3.5. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2 \times 2}$ with d_4 regular. With the notations above, if $e d_2 = d_3 f = 0$ and $a^{\parallel s}$ exists, then $A^{\parallel D}$ exists if and only if $\xi = \beta - \alpha a^{\parallel s} (a d_3 d_4^+ + c)$ is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\parallel s}) & \xi^{-1} d_4 \end{bmatrix}$, where

$$\begin{aligned} u &= s a + 1 - s s^+, \\ \alpha &= d_2 a + d_4 b, \\ \beta &= \alpha d_3 d_4^+ + d_2 c + d_4 d + e, \\ \xi &= \beta - \alpha a^{\parallel s} (a d_3 d_4^+ + c), \\ x_1 &= [(1 - a^{\parallel s} a) d_3 d_4^+ - a^{\parallel s} c] \xi^{-1}, \\ x_2 &= u^{-1} - x_1 \alpha u^{-1}. \end{aligned}$$

Question 3.6. Given a regular matrix D , can we give further equivalent conditions such that $A^{\parallel D}$ exists without additional conditions ?

ACKNOWLEDGMENTS

The first author is grateful to China Scholarship Council for supporting him to pursue his further study in University of Minho, Portugal. This research is supported by the FEDER Funds through Programa Operacional

Factores de Competitividade-COMPETE', the Portuguese Funds through FCT- 'Fundação para a Ciência e Tecnologia', within the project PEst-OE/MAT/UI0013/2014, the National Natural Science Foundation of China (No. 11201063 and No. 11371089), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120092110020), the Natural Science Foundation of Jiangsu Province (No. BK20141327), the Foundation of Graduate Innovation Program of Jiangsu Province (No. CXLX13-072) and the Fundamental Research Funds for the Central Universities (No. 22420135011).

References

References

- [1] J.A. Green, On the structure of semigroups, *Ann. Math.* 54 (1951) 163-172.
- [2] N. Jacobson, The radical and semi-simplicity for arbitrary rings, *Amer. J. Math.* 67 (1945) 300-320.
- [3] X. Mary, Natural generalized inverse and core of an element in semigroups, rings and Banach and operator algebras, *Eur. J. Pure Appl. Math.* 6 (2013) 413-427.
- [4] X. Mary, On generalized inverses and Green's relations, *Linear Algebra Appl.* 434 (2011) 1836-1844.
- [5] X. Mary, P. Patrício, Generalized inverses modulo \mathcal{H} in semigroups and rings, *Linear Multilinear Algebra* 61 (2013) 886-891.
- [6] X. Mary, P. Patrício, The inverse along a lower triangular matrix, *Appl. Math. Comput.* 219 (2012) 1130-1135.
- [7] P. Patrício, R. Puystjens, About the von Neumann regularity of triangular block matrices, *Linear Algebra Appl.* 332/334 (2001) 485-502.
- [8] P. Patrício, R. Puystjens, Generalized invertibility in two semigroups of a ring, *Linear Algebra Appl.* 377 (2004) 125-139.
- [9] D.S. Rakić, N.C. Dinčić and D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, *Linear Algebra Appl.* 463 (2014) 115-133.